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Correlation Inequalities and Contour Estimates

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We give a simple estimate on the probability of contours in classical ferromagnetic spin systems, based on Griffiths' or Ginibre's correlation inequalities. This includes quite general one- and two-component spin models. Some extension also holds for all n-component anisotropic or isotropic rotators.

KEY WORDS: Classical spin systems; phase transition; Peierls' argument.

1. INTRODUCTION

In the Peierls' argument for phase transition there are two distinct steps: one is an entropy estimate which counts the number of contours of a given length and the other is an energy estimate. Take, for example, the Ising model in any dimension:

$$-H = J\sum_{\langle ij\rangle}\sigma_i\sigma_j, \qquad \sigma_i = \pm 1$$

Then for any finite volume Λ and any contour γ in Λ, the energy estimate is

$$P_\Lambda(\sigma_i\sigma_j = -1 \quad \forall\langle ij\rangle \in \gamma) \leqslant \exp(-2\beta J|\gamma|) \tag{1}$$

where P_Λ is the probability in the Gibbs state in Λ with ferromagnetic boundary conditions (e.g., + or free) on Λ.

Inequalities like (1) are obtained by estimating, for each configuration where the contour γ occurs, the gain in energy produced by "flipping" the

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spins inside γ. This works well for discrete spins but if the spins are continuous (as in rotator models) one may find some configurations where flipping the spins produces an arbitrarily small gain in energy. However, the method was extended to continuous spins by Bortz and Griffiths,(1) van Beijeren and Sylvester,(2) and in a quite general way by Malyshev.(3) These authors suitably modify the definition of contours in order to extract a fixed energy gain (at least for models with a finite number of ground states). In particular, Malyshev(3) was able to prove the existence of phase transition for anisotropic rotators with any anisotropy on a lattice of dimension greater than or equal to 2.

Let us mention that another method allows one to prove phase transition for continuous spins. By correlation inequalities, one compares these models with other continuous or discrete spin models that one can handle directly. This was initiated by Nelson,(4) extended by van Beijeren and Sylvester(2) to all one-component continuous spins with even single-spin measure and pair interactions, and by Kunz, Pfister, and Vuillermot(5) to two-component anisotropic rotators. Finally, a general form of such comparison inequalities was proven by Wells.[4]

Coming back to contour estimates, a new method was developed in Refs. 8–10 based on reflection positivity. This method does not work configuration by configuration but reduces estimates like (1) to some thermodynamic estimate on the free energy. It allows one to deal with some nonferromagnetic models, also with quantum models and n-component anisotropic rotators.(9)

In the method presented here, we also avoid making an estimate on each configuration. However, we go in a direction opposite to the one of the reflection positivity method. Instead of reducing everything to a thermodynamic estimate, we reduce an estimate like (1), which has to be uniform in the volume Λ, to an estimate on a fixed finite set of sites. And there it is a simple computation. Because this reduction is based on the correlation inequalities of Griffiths(11) or Ginibre,(12) we are limited to ferromagnetic systems but not to reflection positive ones.

In Section 2, we state and prove our results for one-component systems with general ferromagnetic interactions and even single-spin measures.

In Section 3, we extend these results to plane rotators (two-component models) and prove the result quoted in Ref. 13.

Section 4 deals with n-component rotators, where somewhat weaker results are obtained; however, this also proves a contour estimate which implies phase transition for n-component anisotropic rotators.

[4] See Ref. 6 or, for a published version, Ref. 7.

2. ONE-COMPONENT SYSTEMS

Consider the probability measure μ on $\mathbb{R}^{|\Lambda|}$, where Λ is some finite subset of $\mathbb{Z}^d$:

$$d\mu = Z^{-1}\exp\left[\beta \sum_{A\subset\Lambda} J(A)S_A\right]\prod_{i\in\Lambda} d\nu_i(S_i)$$

where the sum $\sum_A$ runs over multiplicity functions in Λ (i.e., functions from Λ into $\mathbb{N}$), $S_A = \prod_i S_i^{A(i)}$; $J(A) \geqslant 0$ and $\beta = T^{-1}$ is the inverse temperature; the measures ν_i on $\mathbb{R}$ are even and decay sufficiently rapidly at infinity; and

$$Z = \int_{\mathbb{R}^{|\Lambda|}} \exp\left[\beta \sum_{A\subset\Lambda} J(A)S_A\right]\prod_{i\in\Lambda} d\nu_i(S_i)$$

Let $\langle\,\rangle$ denote the expectation with respect to μ.

Define, for any multiplicity function A,

$$C_A = \frac{\int S_A \exp\left[\frac{1}{2}\beta J(A)S_A\right]\prod_i d\nu_i(S_i)}{\int \exp\left[\frac{1}{2}\beta J(A)S_A\right]\prod_i d\nu_i(S_i)}$$

Notice that $C_A \geqslant 0$ is increasing in β and that, as $\beta\to\infty$, $C_A \to \prod_i[\sup(\operatorname{supp}\nu_i)]^{A(i)}$ (which may be infinity) if $J(A)\neq 0$.

Theorem 1. For any family $\mathfrak{M}$ of multiplicity functions and any $d_A > 0$, $A\in\mathfrak{M}$,

$$\left\langle \prod_{A\in\mathfrak{M}} \chi(S_A \leqslant C_A - d_A)\right\rangle \leqslant \exp\left[-\frac{\beta}{2}\sum_{A\in\mathfrak{M}} J(A)d_A\right]$$

In particular ($d_A = C_A$),

$$\left\langle \prod_{A\in\mathfrak{M}} \chi(S_A \leqslant 0)\right\rangle \leqslant \exp\left[-\frac{\beta}{2}\sum_{A\in\mathfrak{M}} J(A)C_A\right]$$

$\chi(\cdot)$ being the characteristic function of the event in parentheses.

Proof. We write

$$\begin{aligned}
&\left\langle \prod_{A\in\mathfrak{M}} \chi(S_A \leqslant C_A - d_A)\right\rangle \\
&\quad= \left\langle \prod_{A\in\mathfrak{M}}\left\{\chi(S_A \leqslant C_A - d_A)\exp\left[\frac{\beta}{2}J(A)S_A\right]\exp\left[-\frac{\beta}{2}J(A)S_A\right]\right\}\right\rangle \\
&\quad\leqslant \exp\frac{\beta}{2}\sum_{A\in\mathfrak{M}} J(A)(C_A - d_A)\left\langle \prod_{A\in\mathfrak{M}}\exp\left[-\frac{\beta}{2}J(A)S_A\right]\right\rangle \qquad (2)
\end{aligned}$$

But

$$\left\langle \prod_{A\in\mathfrak{M}} \exp\left[-\frac{\beta}{2}J(A)S_A\right]\right\rangle = \left[\left\langle \prod_{A\in\mathfrak{M}} \exp\frac{\beta}{2}J(A)S_A\right\rangle'\right]^{-1} \tag{3}$$

where $\langle\ \rangle'$ is the same measure as $\langle\ \rangle$ but with $J(A)$ replaced by $J(A)/2$ for all $A \in \mathfrak{M}$.

Now, by Jensen's inequality

$$\left\langle \prod_{A\in\mathfrak{M}} \exp\frac{\beta}{2}J(A)S_A\right\rangle' \geqslant \exp\frac{\beta}{2}\sum_{A\in\mathfrak{M}} J(A)\langle S_A\rangle' \tag{4}$$

and by Griffiths' inequalities[11] (monotonicity of $\langle S_A\rangle'$ in all the J's)

$$\langle S_A\rangle' \geqslant C_A \tag{5}$$

Combining (2), (3), (4), (5) ends the proof. ■

Remarks. (1) We use correlation inequalities only to get a lower bound on $\langle S_A\rangle'$. However, the proof shows that if, by some means, we know that $\langle S_A\rangle'$ remains strictly positive as $\Lambda\to\mathbb{Z}^d$, $\beta\to\infty$, then by Jensen's inequality the probability that S_A be negative is exponentially small with β as $\beta\to\infty$.

(2) Given Theorem 1, the question of whether there is a phase transition or not for β large depends only on the "geometry" of the set of A's where $J(A)\neq 0$. For the spin-1/2 case $[\nu_i(S_i)=\delta(S_i^2-1)]$, it is shown in Ref. 14 how to solve that problem by purely algebraic methods. Although in Ref. 14 the "decomposition property" is also used in the energy estimate, Theorem 1 shows that it only plays a crucial role in the entropy estimate. Using the results of Ref. 6 one can extend the analysis of Ref. 14 to general even a priori measures; see Ref. 7.

3. PLANE ROTATOR MODELS

Let $\phi_i \in S^1 = [-\pi,\pi]$ and the probability measure μ on $(S^1)^{|\Lambda|}$ given by

$$d\mu = Z^{-1}\exp\left[\beta\sum_{(i,j)\in\Lambda} J_{ij}\cos(\phi_i-\phi_j) + \sum_{i\in\Lambda} h_i\cos\phi_i\right]\prod_{i\in\Lambda} d\phi_i$$

where the sum is over all pairs of points $i,j\in\Lambda$ and $J_{ij}\geqslant 0$, $h_i\geqslant 0$. (h_i allows for ferromagnetic boundary conditions). Again $\langle\,\rangle$ is the expectation with respect to μ.

Theorem 2. There exists a constant c such that, for any $d>0$ and any family $\mathfrak{M}$ of pairs (i,j) $i,j\in\Lambda$:

$$\left\langle \prod_{(i,j)\in\mathfrak{M}} \chi(|\phi_i-\phi_j| \geqslant d \,\mathrm{modulo}\, 2\pi)\right\rangle \leqslant \left[c\exp\left(-\frac{\beta d^2}{\pi^2}\right)\right]^{\sum_{(i,j)\in\mathfrak{M}} J_{ij}}$$

Proof. As in the preceding proof, we write

$$\left\langle \prod_{(i,j)\in\mathfrak{M}} \chi(|\phi_i - \phi_j| > d \bmod 2\pi)\right\rangle$$

$$= \left\langle \prod_{(i,j)\in\mathfrak{M}} \chi(|\phi_i - \phi_j| > d \bmod 2\pi) \exp\left[\frac{\beta}{2} J_{ij}\cos(\phi_i - \phi_j)\right]\right.$$

$$\left. \times \exp\left[-\frac{\beta}{2} J_{ij}\cos(\phi_i - \phi_j)\right]\right\rangle$$

$$\leqslant \exp\left[\frac{\beta}{2}\left(1 - \frac{2}{\pi^2}d^2\right)\sum_{(i,j)\in\mathfrak{M}} J_{ij}\right]\left\langle \prod_{(i,j)\in\mathfrak{M}} \exp\left[-\frac{\beta}{2} J_{ij}\cos(\phi_i - \phi_j)\right]\right\rangle$$

because $\cos x \leqslant 1 - (2/\pi^2)d^2$ for $d \leqslant |x| \leqslant \pi$.

Now, by inverting the last factor and using Jensen's and Ginibre's inequalities[12] as in the proof of Theorem 1, we obtain

$$\left\langle \prod_{(i,j)\in\mathfrak{M}} \exp\left[-\frac{\beta}{2} J_{ij}\cos(\phi_i - \phi_j)\right]\right\rangle \leqslant \exp\left(-\frac{\beta}{2}\sum_{(i,j)\in\mathfrak{M}} J_{ij} c_{ij}\right)$$

where

$$c_{ij} = \frac{\int_{-\pi}^{\pi}\cos(\phi_i - \phi_j)\exp\left[\frac{1}{2}\beta J_{ij}\cos(\phi_i - \phi_j)\right] d\phi_i\, d\phi_j}{\int_{-\pi}^{\pi}\exp\left[\frac{1}{2}\beta J_{ij}\cos(\phi_i - \phi_j)\right] d\phi_i\, d\phi_j}$$

But by doing an asymptotic expansion of c_{ij} one sees that $c_{ij} \geqslant 1 - c'T$ which concludes the proof. [In the theorem $c = \exp(+\frac{1}{2}\beta c'T)$.] ■

Remarks. (1) The result extends to any interaction

$$-H = \sum_{m\subset\Lambda} J(m)\cos m\phi, \qquad J(m) \geqslant 0$$

because it satisfies Ginibre's inequality[12]; in that case one obtains

$$\left\langle \prod_{m\in\mathfrak{M}} \chi(|m\phi| > d \bmod 2\pi)\right\rangle \leqslant \left[c_{\mathfrak{M}} \exp\left(-\frac{\beta d^2}{\pi^2}\right)\right]^{\sum_{m\in\mathfrak{M}} J(m)}$$

(2) Moreover, for anisotropic rotators:

$$-H = \sum_{i,j\in\Lambda} J_{ij}\cos(\phi_i - \phi_j) + D_{ij}\cos\phi_i\cos\phi_j$$

$$J_{ij} \geqslant 0, \qquad D_{ij} \geqslant 0$$

We have, using the proof of Theorem 1 and Ginibre's inequality, that, for any $\epsilon > 0$,

$$\left\langle \prod_{(i,j)\in\mathfrak{M}} \chi(\cos\phi_i\cos\phi_j \leqslant 1 - \epsilon)\right\rangle \leqslant \left(c \exp - \frac{\beta\epsilon}{2}\right)^{\sum_{(i,j)\in\mathfrak{M}} D_{ij}}$$

(3) Instead of having rotators with fixed length one can allow for some other a priori measure (see Ref. 16).

4. *N*-COMPONENT ROTATORS

Let for each $i \in \Lambda$ $\mathbf{S}_i \in S^{n-1}$ the unit sphere in $\mathbb{R}^n$,

$$\mathbf{S}_i = (S_i^1, \ldots, S_i^n) \quad \text{and} \quad |\mathbf{S}_i|^2 = \sum_{\alpha=1}^{n} (S_i^\alpha)^2 = 1$$

Consider the probability measure μ on $(S^{n-1})^{|\Lambda|}$:

$$d\mu = Z^{-1}\exp\left[\sum_{i,j\in\Lambda}(J_{ij}\mathbf{S}_i \cdot \mathbf{S}_j + D_{ij}S_i^1 S_j^1) + \sum_{i\in\Lambda} h_i S_i^1\right]\prod_{i\in\Lambda}\delta(|\mathbf{S}_i|^2 - 1)\,d\mathbf{S}_i$$

with $J_{ij} \geqslant 0$, $D_{ij} \geqslant 0$, $h_i \geqslant 0$.

$\mathbf{S}_i \cdot \mathbf{S}_j$ is the scalar product between $\mathbf{S}_i$ and $\mathbf{S}_j$. $J_{ij}\mathbf{S}_i\mathbf{S}_j$ is the isotropic part of the interaction and D_{ij} is the anisotropy favoring the direction S_i^1; h_i allows for ferromagnetic boundary conditions in that direction.

Let $\langle\,\rangle$ be the expectation with respect to μ.

Theorem 3. There exists a constant c such that, for any family $\mathfrak{M}$ of pairs (i, j), $i, j \in \Lambda$:

(a) $$\left\langle \prod_{(i,j)\in\mathfrak{M}} \chi(S_i^1 S_j^1 \leqslant 0)\right\rangle \leqslant \left[c\exp\left(-\frac{\beta}{2en^2}\right)\right]^{\sum_{(i,j)\in\mathfrak{M}} D_{ij}}$$

(b) $$\left\langle \prod_{(i,j)\in\mathfrak{M}} \mathbf{S}_i \cdot \mathbf{S}_j \leqslant 0\right\rangle \leqslant \left[c\exp\left(-\frac{\beta}{2en^2}\right)\right]^{\sum_{(i,j)\in\mathfrak{M}} J_{ij}}$$

Proof. (a) Following the proof of Theorem 1, we see that we have only to obtain a lower bound, uniform in Λ and β, on $\langle S_i^1 S_j^1\rangle'$ for those pairs (i, j) with $D_{ij} > 0$. $\langle\,\rangle'$ corresponds to the measure similar to $\langle\,\rangle$ but with $D_{ij}/2$ instead of D_{ij} for all pairs $(i, j) \in \mathfrak{M}$.

We derive this lower bound in two steps: first we show that it is enough to bound from below $\langle (S_i^1)^2 (S_j^1)^2\rangle'$ and then we use an inequality of Simon,[15] valid for all n-component rotators, to bound the latter quantity.

(1) We start by conditioning in $\langle S_i^1 S_j^1\rangle'$ on all the spin components other than the first one:

$$\langle S_i^1 S_j^1\rangle' = \int \langle S_i^1 S_j^1\rangle(\{S_i^\alpha\}_{i\in\Lambda,\alpha=2,\ldots,n})\,d\rho(S_i^\alpha) \tag{6}$$

where $\langle\,\rangle(\{S_i^\alpha\}_{\alpha=2,i\in\Lambda}^n)$ is the conditional expectation with respect to $\{S_i^\alpha\}$,

$\alpha \neq 1$, or, explicitly, the expectation with respect to

$$Z^{-1}(\{S_i^\alpha\})\exp\left\{\beta \sum_{(i,j)\in\Lambda}\left[J_{ij}+(\tfrac{1}{2})D_{ij}\right]S_i^1S_j^1\right\}$$
$$\times \prod_{i\in\Lambda}\delta\left\{\left(S_i^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_i^\alpha)^2\right]\right\}dS_i^1 \tag{7}$$

Z being the corresponding normalization. The $(\frac{1}{2})$ multiplying D_{ij} indicates that, for those pairs $(i,j)\in\mathfrak{M}$, D_{ij} is replaced by $D_{ij}/2$.

$\rho(S_i^\alpha)$ is simply the measure $\langle\,\rangle'$ restricted to $\{S_i^\alpha\}$, $\alpha\neq 1$.

Notice that, for any value of $\{S_i^\alpha\}$, $\alpha\neq 1$, (7) is a one-component ferromagnetic measure and therefore satisfies Griffiths' second inequality.[11] Thus we have

$$\langle S_iS_j\rangle\left(\{S_i^\alpha\}_{\alpha=1}^n, i\in\Lambda\right)\geqslant\langle S_iS_j\rangle\left(S_i^\alpha,S_j^\alpha,\alpha\neq 1\right) \tag{8}$$

where the right-hand side is the expectation with respect to

$$Z^{-1}(S_i^\alpha,S_j^\alpha,\alpha\neq 1)\exp\left[\beta\left(J_{ij}+\frac{D_{ij}}{2}\right)S_i^1S_j^1\right]$$
$$\times\delta\left\{\left(S_i^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_i^\alpha)^2\right]\right\}\delta\left\{\left(S_j^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_j^\alpha)^2\right]\right\}dS_i^1\,dS_j^1$$

Now, $\langle S_iS_j\rangle(S_i^\alpha,S_j^\alpha,\alpha\neq 1)$ is larger, again by Griffiths' inequalities[11] than its value when $J_{ij}=0$ and $\beta D_{ij}/2=1$ at least for $\beta D_{ij}/2\geqslant 1$. (For $\beta D_{ij}/2\leqslant 1$ the inequality in the theorem is trivial to prove if we choose c appropriately.) But, if $\beta D_{ij}/2=1$, and $J_{ij}=0, Z(S_i^\alpha,S_j^\alpha,\alpha\neq 1)\leqslant e$ independently of $S_i^\alpha,S_j^\alpha,\alpha\neq 1$. Now we expand the exponential in

$$\int S_i^1S_j^1\exp\left(\beta\frac{D_{ij}}{2}S_i^1S_j^1\right)\delta\left\{\left(S_i^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_i^\alpha)^2\right]\right\}$$
$$\times\delta\left\{\left(S_j^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_j^\alpha)^2\right]\right\}dS_i^1\,dS_j^1$$

for $\beta D_{ij}/2=1$, and using the positivity of all terms, keep only the one of first degree, so that

$$\langle S_i^1S_j^1\rangle(S_i^\alpha,S_j^\alpha,\alpha\neq 1)\geqslant e^{-1}\int\left(S_i^1\right)^2\left(S_j^1\right)^2\delta\left\{\left(S_i^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_i^\alpha)^2\right]\right\}$$
$$\times\delta\left\{\left(S_j^1\right)^2-\left[1-\sum_{\alpha=2}^{n}(S_j^\alpha)^2\right]\right\}dS_i^1\,dS_j^1 \tag{9}$$

and the last integral is

$$\left[1-\sum_{\alpha=2}^{n}(S_i^\alpha)^2\right]\left[1-\sum_{\alpha=1}^{n}(S_j^\alpha)^2\right] \tag{10}$$

Inserting (8), (9), (10) in (6) we have (for $\beta D_{ij}/2 \geqslant 1$)

$$\langle S_i^1 S_j^1\rangle' \geqslant e^{-1}\int\left[1-\sum_{\alpha=2}^{n}(S_i^\alpha)^2\right]\left[1-\sum_{\alpha=1}^{n}(S_j^\alpha)^2\right]d\rho(S_i^\alpha) \tag{11}$$

But because in $d\rho$ the $\sum_{\alpha=2}^{n}(S_i^\alpha)^2$ is constrained by the δ measures to be equal to $1-(S_i^1)^2$ and because $d\rho$ is nothing but the measure $\langle\,\rangle'$ restricted to the variables $\{S_i^\alpha\}$ $\alpha \neq 1$, the right-hand side of (11) is equal to $\langle (S_i^1)^2(S_j^1)^2\rangle'$ and therefore we have shown that

$$\langle S_i^1 S_j^1\rangle \geqslant e^{-1}\langle (S_i^1)^2(S_j^1)^2\rangle'$$

(2) Now we have only to bound $\langle (S_i^1)^2(S_j^1)^2\rangle'$ from below. But this follows easily from an inequality of Simon,[15] which says that

$$\left\langle (S_i^1)^2\left[(S_j^1)^2-(S_j^\alpha)^2\right]\right\rangle' \geqslant 0 \tag{12}$$

and

$$\langle (S_i^1)^2-(S_i^\alpha)^2\rangle' \geqslant 0 \tag{13}$$

for all $\alpha = 1, \ldots, n$. So that

$$\begin{aligned}
\langle (S_i^1)^2(S_j^1)^2\rangle &\geqslant \frac{1}{n}\sum_{\alpha=1}^{n}\langle (S_i^1)^2(S_j^\alpha)^2\rangle \qquad \text{by (12)}\\
&= \frac{1}{n}\langle (S_i^1)^2\rangle \qquad \text{because } \sum_{\alpha=1}^{n}(S_j^\alpha)^2 = 1\\
&\geqslant \frac{1}{n^2}\left\langle \sum_{\alpha=1}^{n}(S_i^\alpha)^2\right\rangle \qquad \text{by (13)}\\
&= \frac{1}{n^2}
\end{aligned}$$

(b) We can reduce ourselves to obtain a lower bound on

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j\rangle'$$

where in $\langle\,\rangle'$ J_{ij} is replaced by $J_{ij}/2$ for all $(i,j) \in \mathfrak{M}$.

But since $\langle S_i^\alpha S_j^\alpha\rangle'$ is positive by the first Griffiths inequality (valid for all n-component rotators[5,16]) we can repeat the argument given in the proof of (a) for $\langle S_i^1 S_j^1\rangle$. Actually if $D_{ij} = h_i = 0$, we gain a factor of n because $\langle S_i^\alpha S_j^\alpha\rangle = \langle S_i^1 S_j^1\rangle \forall\alpha$. ■

Remarks. (1) Theorem 3 is weaker than the analog of Theorems 1 and 2 because we do not show that $S_i^1 S_j^1$ or $\mathbf{S}_i \cdot \mathbf{S}_j$ approaches 1 as $\beta \to \infty$ but only that it cannot be negative. Actually one could push the estimates up to $S_i^1 S_j^1 \leqslant 1/2en^2 - \epsilon$, $\epsilon > 0$, but for the case of reflection positive interactions, one can show that the probability that $\mathbf{S}_i \cdot \mathbf{S}_j \leqslant 1 - \epsilon$ is exponentially small as $\beta \to \infty$, for any $\epsilon > 0$.

(2) However, Theorem 3 (a) is all that is needed in order to prove that anisotropic rotators with any anisotropy [that is, $D_{ij} \neq 0$ but as small as one wishes for (i, j) nearest neighbors] have a spontaneous magnetization for β large when the lattice dimension is greater than or equal to 2.

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